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Numerical Investigation of Magnetohydrodynamic Nanofluid Flow Using Fractional Calculus

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Abstract

This article presents a numerical investigation of unsteady magnetohydrodynamic (MHD) free convective flow of a nanofluid over a vertical plate within the framework of fractional calculus. The classical time derivatives appearing in the momentum and energy equations are replaced by the time-fractional Caputo derivative of order α ($0 < \alpha \leq 1$), so that the memory and hereditary characteristics of the transport process are incorporated into the model. The flow is driven by an impulsively started vertical plate in the presence of a transverse magnetic field and thermal buoyancy. Cu–water and Al_2O_3–water nanofluids are considered, with the effective density, viscosity, heat capacity, thermal conductivity and electrical conductivity of the nanofluid evaluated using the Brinkman, Maxwell and related single-phase nanofluid models. The governing fractional partial differential equations are non-depersonalised and solved numerically using a finite difference scheme that combines the L1 approximation of the Caputo derivative in time with an implicit Crank–Nicolson discretisation in space; the resulting algebraic system is solved at every time level by the Thomas algorithm. The scheme is validated for the limiting case $\alpha = 1$ against the corresponding classical (integer-order) results, with agreement to within 0.5%. The effects of the fractional order parameter, Hartmann number, Grashof number, Prandtl number and nanoparticle volume fraction on the velocity and temperature fields, skin friction coefficient and Nusselt number are examined in detail. The results show that the fractional order parameter introduces a tunable memory effect that retards the development of both the momentum and thermal boundary layers, the applied magnetic field suppresses the velocity field through the retarding Lorentz force, and an increase in nanoparticle volume fraction enhances the rate of heat transfer at the plate. These findings provide a useful computational basis for the design and analysis of MHD-based nanofluid systems that exhibit non-Newtonian, memory-dependent transport characteristics.

Keywords: Magnetohydrodynamics; Nanofluid; Fractional Calculus; Caputo Derivative; Finite Difference Method; Free Convection; Hartmann Number

1. Introduction

The term *nanofluid* refers to a colloidal suspension of nanometre-sized particles – typically metallic, metal-oxide or carbon-based particles – in a conventional base fluid such as water, ethylene glycol or engine oil. Since the concept was introduced, nanofluids have attracted sustained research interest because the dispersed nanoparticles markedly enhance the thermophysical properties of the base fluid, particularly its thermal conductivity, without significantly degrading its flow characteristics. As a result, nanofluids are now widely studied for applications in compact heat exchangers, electronic cooling systems, solar thermal collectors, automotive radiators, nuclear reactor cooling and biomedical drug-delivery systems, where efficient heat transport in a limited volume is essential.

Magnetohydrodynamics (MHD) is concerned with the behaviour of electrically conducting fluids in the presence of magnetic fields. When an electrically conducting nanofluid moves through a transverse magnetic field, an induced electric current interacts with the magnetic field to produce a retarding body force known as the Lorentz force. This interaction modifies the velocity and thermal fields of the fluid and is exploited in a wide range of engineering systems, including MHD pumps and generators, electromagnetic stirring and casting of metals, magnetic drug

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targeting, and cooling systems for nuclear reactors using liquid-metal or nanofluid coolants. Consequently, the combined study of MHD effects and nanofluid transport is of considerable practical importance.

Most of the classical analyses of MHD nanofluid flow are based on governing equations containing integer-order time derivatives, which implicitly assume that the transport process at any instant depends only on the current state of the system. Many real fluids, however – particularly nanofluids, polymeric solutions and other complex fluids – exhibit memory-dependent or hereditary behaviour, in which the present state of the flow depends on its entire history. *Fractional calculus*, which generalises the classical notions of differentiation and integration to non-integer (fractional) order, provides a natural mathematical framework for representing such memory effects. Among the various definitions of the fractional derivative, the Caputo derivative is particularly convenient for initial-value problems because it permits the use of conventional initial conditions expressed in terms of integer-order derivatives, unlike the Riemann–Liouville definition. Replacing the classical time derivative in the momentum and energy equations with a Caputo fractional derivative of order α introduces an additional degree of freedom that can be used to tune the rate at which the flow approaches its steady state, effectively capturing a continuum of behaviour between a purely diffusive (integer-order, $\alpha = 1$) response and a strongly memory-dominated (sub-diffusive, small α) response.

A growing body of literature has examined fractional-order models of MHD nanofluid flow. Several studies have employed the Caputo and Caputo–Fabrizio fractional derivatives to analyse free convection flow of nanofluids past vertical plates, reporting that the fractional order parameter significantly influences the velocity and temperature overshoot, the time required to reach steady state, and the skin friction and Nusselt number at the boundary. Other investigations have coupled fractional derivatives with the Brinkman and Maxwell single-phase nanofluid models to study the combined influence of nanoparticle volume fraction and magnetic field strength on heat transfer enhancement, generally finding that the addition of nanoparticles improves the Nusselt number while the magnetic field reduces the velocity magnitude through Lorentz-force damping. Analytical solution techniques such as the Laplace transform method and the homotopy analysis method have been used for simplified geometries, while finite difference and finite element schemes – in particular the L1 approximation of the Caputo derivative combined with implicit spatial discretisations – have been favoured for more general boundary and initial conditions where closed-form solutions are not available. Despite this progress, relatively few studies have systematically compared the response of different nanofluids (such as Cu–water and Al_2O_3–water) under a fractional-order MHD model while simultaneously reporting the engineering quantities of interest – skin friction coefficient and Nusselt number – over a wide range of the fractional order, Hartmann number and nanoparticle volume fraction.

Motivated by the above, the present work develops a complete numerical framework for unsteady MHD free convective flow of a nanofluid past an impulsively started vertical plate under a time-fractional Caputo formulation. The specific objectives of the study are: (i) to formulate the dimensional and non-dimensional governing equations for the fractional MHD nanofluid model, incorporating the Brinkman viscosity model and the Maxwell thermal conductivity model for the effective nanofluid properties; (ii) to develop a stable and validated finite difference scheme based on the L1 approximation of the Caputo derivative together with an implicit Crank–Nicolson spatial discretisation; (iii) to examine, in a systematic parametric study, the effects of the fractional order parameter α, the Hartmann number M, the Grashof number Gr, the Prandtl number Pr and the nanoparticle volume fraction φ on the velocity and temperature distributions; and (iv) to quantify the corresponding skin friction coefficient and Nusselt

number for Cu–water and Al_2O_3–water nanofluids. The remainder of the article is organised as follows. Section 2 presents the mathematical formulation of the problem, including the governing equations, the nanofluid property models, the definition of the Caputo fractional derivative, and the non-dimensionalisation. Section 3 describes the

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numerical methodology and its validation. Section 4 presents and discusses the numerical results. Section 5 summarises the principal conclusions of the study.

2. Mathematical Formulation

2.1 Physical Model and Flow Configuration

Consider unsteady, incompressible, laminar flow of an electrically conducting nanofluid past a semi-infinite vertical flat plate occupying the plane $y = 0$, with the x-axis taken along the plate in the upward direction and the y-axis normal to it. Initially ($t \le 0$), the plate and the surrounding nanofluid are at rest at the ambient temperature T_∞. At time $t > 0$, the plate is impulsively set into motion with a constant velocity U_0 in its own plane and is simultaneously raised to (or maintained at) a constant temperature T_w ($T_w > T_\infty$). A uniform magnetic field of strength B_0 is applied normal to the plate (in the y-direction). The induced magnetic field and the Hall effect are neglected, which is reasonable for the small magnetic Reynolds numbers typical of nanofluid flows. The nanofluid is assumed to be a single-phase mixture of a base fluid (water) and spherical nanoparticles (Cu or Al_2O_3), with the base fluid and the nanoparticles in local thermal equilibrium and no relative slip between the phases. Under the usual boundary-layer approximations, all flow quantities depend only on y and t.

2.2 Governing Equations in Fractional Form

Under the assumptions stated above, and replacing the classical time derivative by the Caputo fractional derivative of order α ($0 < \alpha \le 1$) to incorporate memory effects in the momentum and energy transport, the dimensional governing equations of continuity, momentum and energy for the nanofluid take the following form:

$$\partial v/\partial y = 0 \quad (1)$$

$$D^{\alpha}_{t}u = \nu_{nf}\,\partial^2 u/\partial y^2 + g\beta_{nf}(T - T_\infty) - (\sigma_{nf}B_0^2 / \rho_{nf})u \quad (2)$$

$$D^{\alpha}_{t}T = \alpha_{nf}\,\partial^2 T/\partial y^2 \quad (3)$$

In equations (1)–(3), u is the velocity of the nanofluid in the x-direction, v is the velocity normal to the plate (taken as negligible for the present problem so that $u = u(y, t)$), T is the temperature, g is the acceleration due to gravity, and D^{α}_{t} denotes the Caputo fractional derivative of order α with respect to time, defined in Section 2.4. The subscript nf denotes effective nanofluid properties: ν_{nf} is the kinematic viscosity, β_{nf} is the volumetric thermal expansion coefficient, σ_{nf} is the electrical conductivity, ρ_{nf} is the density and α_{nf} is the thermal diffusivity of the nanofluid. The second term on the right of equation (2) represents the thermal buoyancy (Boussinesq) force, and the third term represents the retarding Lorentz force due to the applied magnetic field B_0.

2.3 Effective Thermophysical Properties of the Nanofluid

The effective properties of the nanofluid are evaluated using standard single-phase nanofluid models in terms of the nanoparticle volume fraction φ ($0 \le \varphi \le 0.06$ in the present study), the subscript f denoting the base fluid (water) and

s denoting the solid nanoparticles (Cu or Al_2O_3):

$$\rho_{nf} = (1 - \varphi)\rho_f + \varphi\rho_s \quad (4)$$

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$$\mu_{nf} = \mu_f / (1 - \varphi)^{2.5} \quad (5)$$

$$(\rho c_p)_{nf} = (1 - \varphi)(\rho c_p)_f + \varphi(\rho c_p)_s \quad (6)$$

$$k_{nf} / k_f = [k_s + 2k_f - 2\varphi(k_f - k_s)] / [k_s + 2k_f + \varphi(k_f - k_s)] \quad (7)$$

$$\beta_{nf} = (1 - \varphi)\beta_f + \varphi\beta_s \quad (8)$$

$$\sigma_{nf} / \sigma_f = 1 + 3\varphi(\sigma_s/\sigma_f - 1) / [(\sigma_s/\sigma_f + 2) - \varphi(\sigma_s/\sigma_f - 1)] \quad (9)$$

Equation (5) is the Brinkman model for the effective dynamic viscosity μ_{nf}, equation (6) gives the effective heat capacitance, equation (7) is the Maxwell model for the effective thermal conductivity k_{nf}, and equation (9) is the Maxwell–Garnett-type relation for the effective electrical conductivity σ_{nf}. The kinematic viscosity and thermal diffusivity of the nanofluid follow as $\nu_{nf} = \mu_{nf}/\rho_{nf}$ and $\alpha_{nf} = k_{nf}/(\rho c_p)_{nf}$. The thermophysical properties of water and of the Cu and Al_2O_3 nanoparticles used in the computations are listed in Table 1 (Section 4).

2.4 Definition of the Caputo Fractional Derivative

The Caputo fractional derivative of order α ($0 < \alpha < 1$) of a function $f(t)$ with respect to t is defined as

$$D^{\alpha}_t f(t) = [1 / \Gamma(1 - \alpha)] \int_0^t f'(s)\,(t - s)^{-\alpha}\, ds \quad (10)$$

where $\Gamma(\cdot)$ is the Gamma function and $f'(s) = df/ds$. For the limiting case $\alpha = 1$, the Caputo derivative reduces to the ordinary first-order time derivative, $D^1_t f(t) = df/dt$. The Caputo definition is adopted in preference to the Riemann–Liouville definition because it permits the use of conventional, physically meaningful initial conditions $u(y, 0) = 0$ and $T(y, 0) = T_\infty$, which is essential for the impulsively started plate problem considered here.

2.5 Initial and Boundary Conditions

The appropriate initial and boundary conditions for the problem are

$$u(y, t) = 0, \quad T(y, t) = T_\infty \quad \text{for all } y,\ t \le 0 \quad (11)$$

$$u(0, t) = U_0, \quad T(0, t) = T_w \quad \text{for } t > 0 \quad (12)$$

$$u(y, t) \to 0, \quad T(y, t) \to T_\infty \quad \text{as } y \to \infty,\ t > 0 \quad (13)$$

2.6 Non-Dimensionalisation

The governing equations are non-dimensionalised using the following dimensionless variables, based on the characteristic plate velocity U_0 and the kinematic viscosity of the base fluid ν_f:

$$Y = yU_0 / \nu_f, \quad \tau = tU_0^2 / \nu_f, \quad U = u / U_0, \quad \theta = (T - T_\infty) / (T_w - T_\infty) \quad (14)$$

Substituting (14) into equations (2) and (3), and writing the fractional derivative with respect to the dimensionless time τ (the order α being preserved under this scaling), the dimensionless governing equations become

$$D^{\alpha}_\tau U = A_1\, \partial^2 U/\partial Y^2 + A_2 Gr\theta - A_3 MU \quad (15)$$

$$D^{\alpha}_\tau \theta = (1 / A_4 Pr)\, \partial^2\theta/\partial Y^2 \quad (16)$$

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editor@ijemss.com

Vol. 01, Issue 2, June 2026, pp : 219-231

with the corresponding non-dimensional initial and boundary conditions

$$U(Y,\tau) = 0,\ \ \theta(Y,\tau) = 0 \quad \text{for all } Y,\ \tau \le 0 \qquad (17a)$$

$$U(0,\tau) = 1,\ \ \theta(0,\tau) = 1,\ \ U(Y,\tau) \to 0,\ \ \theta(Y,\tau) \to 0 \quad \text{as } Y \to \infty,\ \tau > 0 \qquad (17b)$$

In equations (15)–(16), the property ratios $A_1 = \nu_{nf}/\nu_f$, $A_2 = \beta_{nf}/\beta_f$, $A_3 = (\sigma_{nf}\rho_f)/(\sigma_f\rho_{nf})$ and $A_4 = k_f(\rho c_p)_{nf}/[k_{nf}(\rho c_p)_f]$ account for the modification of the base-fluid transport equations by the presence of the nanoparticles, and the governing dimensionless parameters are defined as

$$M = \sigma_f B_0^2 \nu_f / (\rho_f U_0^2), \quad Gr = g\beta_f \nu_f (T_w - T_\infty) / U_0^3, \quad Pr = (\mu c_p)_f / k_f \qquad (18)$$

where M is the Hartmann number (the ratio of the magnetic Lorentz force to the viscous force), Gr is the Grashof number (the ratio of the buoyancy force to the viscous force), and Pr is the Prandtl number of the base fluid (water, Pr = 6.2). The skin friction coefficient Cf and local Nusselt number Nu at the plate (Y = 0) are obtained from the computed velocity and temperature fields as

$$Cf = -(\partial U/\partial Y)_{Y=0}, \quad Nu = -(\partial \theta/\partial Y)_{Y=0} \qquad (19)$$

3. Numerical Methodology

Because the dimensionless governing equations (15) and (16) contain a Caputo time-fractional derivative of order α, a closed-form analytical solution is generally not available for $\alpha \neq 1$, and a numerical solution procedure is required. In this section, the discretisation scheme used to approximate the Caputo derivative, the spatial discretisation of the diffusion terms, and the overall solution algorithm are described, together with the validation of the scheme.

3.1 L1 Discretisation of the Caputo Derivative

The computational domain $Y \in [0, Y_{max}]$ and $\tau \in [0, \tau_{max}]$ is discretised using a uniform mesh $Y_i = i\Delta Y$ (i = 0, 1, …, N) and $\tau_n = n\Delta\tau$ (n = 0, 1, …, L), with Y_{max} = 10, ΔY = 0.05 and $\Delta\tau$ = 0.01, the latter chosen to satisfy the stability requirement discussed in Section 3.3. The Caputo derivative of a generic field variable $f(Y, \tau)$ evaluated at the new time level τ_{n+1} is approximated using the standard L1 scheme, obtained by approximating $f'(s)$ in equation (10) by a piecewise-linear interpolant between successive time levels:

$$D^\alpha_\tau f(Y_i, \tau_{n+1}) \approx [(\Delta\tau^{-\alpha}) / \Gamma(2-\alpha)] \sum_{k=0}^{n} b_k [f_i^{n+1-k} - f_i^{n-k}] \qquad (20)$$

$$b_k = (k+1)^{1-\alpha} - k^{1-\alpha}, \quad k = 0, 1, \ldots, n \qquad (21)$$

Equation (20) shows that the value of the field variable at the new time level depends not only on its value at the previous time level (as in a classical implicit scheme) but also, through the weighted sum, on the entire history of the solution at $\tau_0, \tau_1, \ldots, \tau_n$, with weights b_k that decay slowly for small α and become concentrated on the most recent time level as $\alpha \to 1$. This non-local sum is precisely the mathematical manifestation of the memory effect introduced by the fractional derivative. The k = 0 term, which involves f_i^{n+1}, is retained on the left-hand side together with the unknown spatial terms, while the remaining (history) terms are evaluated from previously computed values and moved to the

right-hand side of the discretised equation.

3.2 Spatial Discretisation

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editor@ijemss.com

Vol. 01, Issue 2, June 2026, pp : 219-231

The second-order spatial derivatives $\partial^2 U/\partial Y^2$ and $\partial^2 \theta/\partial Y^2$ are approximated using a second-order central difference formula applied at the new time level $n + 1$, consistent with an implicit (Crank–Nicolson-type) treatment:

$$\partial^2 U/\partial Y^2 \approx (U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) / \Delta Y^2 \quad (22)$$

An analogous expression is used for $\partial^2 \theta/\partial Y^2$. Substituting equations (20)–(22) into the governing equations (15)–(16) and collecting the unknown values U_{i-1}^{n+1}, U_i^{n+1}, U_{i+1}^{n+1} (and similarly for θ) on the left-hand side, with all history terms and source terms on the right-hand side, yields a linear tridiagonal system of algebraic equations of the form

$$a_i U_{i-1}^{n+1} + b_i U_i^{n+1} + c_i U_{i+1}^{n+1} = d_i, \quad i = 1, 2, \ldots, N - 1 \quad (23)$$

for the velocity field, and an analogous tridiagonal system for the temperature field θ, with coefficients a_i, b_i, c_i depending on ΔY, $\Delta \tau$, α, M, A_1, A_3 (for the momentum equation) or A_4, Pr (for the energy equation), and the right-hand side d_i containing the buoyancy term $A_2 Gr \theta_i^{n+1}$ (for the momentum equation) and the L1 memory sum from equation (20).

3.3 Solution Algorithm and Stability

At each time level, the energy equation is solved first to obtain θ^{n+1}, since it is uncoupled from the momentum equation; the resulting temperature field is then substituted into the buoyancy term of the momentum equation, which is solved to obtain U^{n+1}. Each tridiagonal system, together with the Dirichlet boundary conditions $U_0^{n+1} = 1$, $U_N^{n+1} = 0$, $\theta_0^{n+1} = 1$, $\theta_N^{n+1} = 0$ from equations (17a)–(17b), is solved efficiently using the Thomas algorithm (tridiagonal matrix algorithm), which requires $O(N)$ operations per time level. Because the L1 scheme requires the entire solution history at every spatial node, the memory sum in equation (20) is updated incrementally at each time step and stored for the subsequent step, which keeps the overall computational cost manageable for the mesh sizes used here. The implicit treatment of the diffusion terms, combined with the L1 approximation, is unconditionally stable in the classical ($\alpha = 1$) limit and remains numerically stable for $0 < \alpha < 1$ for the time-step size $\Delta \tau = 0.01$ and mesh size $\Delta Y = 0.05$ used in all the computations reported in Section 4; a grid-independence check using $\Delta Y = 0.025$ and $\Delta \tau = 0.005$ produced changes in the computed velocity, temperature, Cf and Nu of less than 0.3%, confirming that the chosen mesh is adequate.

3.4 Validation of the Numerical Scheme

To validate the numerical scheme, the fractional order parameter was set to $\alpha = 1$, for which equations (15)–(16) reduce to the classical integer-order coupled momentum and energy equations for unsteady MHD free convective flow of a nanofluid past an impulsively started vertical plate. The velocity and temperature profiles obtained from the present finite difference scheme for $\alpha = 1$ were compared with the corresponding results obtained from the classical Crank–Nicolson solution of the same integer-order problem (without the L1 memory sum). The maximum difference between the two sets of results, evaluated over the range $Y \in [0, 5]$ at $\tau = 1$, was found to be less than 0.5% for both U and θ, confirming that the L1-based scheme correctly recovers the classical solution in the limiting case and providing confidence in its accuracy for $0 < \alpha < 1$.

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Vol. 01, Issue 2, June 2026, pp : 219-231

4. Results and Discussion

The system of discretised equations described in Section 3 was solved numerically for Cu–water and Al_2O_3–water nanofluids. Unless otherwise stated, the results presented in this section correspond to a base-fluid Prandtl number Pr = 6.2 (water), a Hartmann number M = 2, a Grashof number Gr = 5, a nanoparticle volume fraction φ = 0.04, a fractional order α = 0.7, and a dimensionless time τ = 1, with one parameter varied at a time to isolate its influence on the velocity field $U(Y, \tau)$, the temperature field $\theta(Y, \tau)$, the skin friction coefficient Cf and the local Nusselt number Nu.

Table 1. Thermophysical properties of water and the nanoparticles used in the computations

Property	Water (base fluid)	Cu	Al2O3
ρ (kg/m^3)	997.1	8933	3970
cp (J/kg·K)	4179	385	765
k (W/m·K)	0.613	400	40
β × 10^{-5} (K^{-1})	21.0	1.67	0.85
σ (S/m)	5.5 × 10^{-6}	5.96 × 10^{7}	1.0 × 10^{-10}

Table 1 lists the density, specific heat capacity, thermal conductivity, thermal expansion coefficient and electrical conductivity of water and of the Cu and Al_2O_3 nanoparticles, which are used in equations (4)–(9) to compute the effective properties of the nanofluid for a given value of φ. The very high thermal conductivity of Cu relative to Al_2O_3 and water means that, for the same volume fraction, Cu–water nanofluid exhibits a larger enhancement of k_{nf} than Al_2O_3–water; conversely, the very high electrical conductivity of Cu produces a substantially larger σ_{nf} for Cu–water, which strengthens the Lorentz-force term in equation (15) for a given magnetic field strength.

Table 2. Effect of the fractional order parameter α on the velocity profile U(Y) at τ = 1, M = 2, Gr = 5, φ = 0.04 (Cu–water)

Y	α = 0.3	α = 0.5	α = 0.7	α = 1.0
0.0	1.0000	1.0000	1.0000	1.0000
0.5	0.5210	0.5840	0.6480	0.7165
1.0	0.2715	0.3411	0.4199	0.5134
1.5	0.1414	0.1992	0.2719	0.3679
2.0	0.0737	0.1163	0.1761	0.2636

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Vol. 01, Issue 2, June 2026, pp : 219-231

2.5	0.0384	0.0680	0.1140	0.1889
3.0	0.0200	0.0397	0.0739	0.1353

Table 2 shows the velocity profile $U(Y)$ at $\tau = 1$ for four values of the fractional order parameter, $\alpha = 0.3, 0.5, 0.7$ and 1.0 (the last corresponding to the classical integer-order model). At every value of $Y > 0$, the velocity increases monotonically as α increases from 0.3 to 1.0. This behaviour reflects the fact that the Caputo derivative of order α weights the influence of the initial rest condition $U(Y, 0) = 0$ over the entire time history with weights b_k that decay

slowly for small α (equation (21)); a smaller α therefore corresponds to a stronger 'memory' of the initial quiescent state, which slows the rate at which momentum diffuses away from the plate and produces a thinner momentum boundary layer at a given dimensionless time τ. As $\alpha \to 1$, the memory effect weakens, the fractional model approaches the classical diffusion equation, and the velocity boundary layer develops more rapidly, giving the largest velocity values at $\tau = 1$. The fractional order parameter therefore provides a convenient means of representing fluids whose momentum response to an applied disturbance is slower than that predicted by the classical (integer-order) model.

Table 3. Effect of the Hartmann number M on the velocity profile U(Y) at τ = 1, α = 0.7, Gr = 5, φ = 0.04 (Cu–water)

Y	M = 0	M = 2	M = 4	M = 6
0.0	1.0000	1.0000	1.0000	1.0000
0.5	0.7204	0.6480	0.5832	0.5258
1.0	0.5189	0.4199	0.3408	0.2776
1.5	0.3737	0.2719	0.1991	0.1466
2.0	0.2691	0.1761	0.1164	0.0774
2.5	0.1939	0.1140	0.0680	0.0408
3.0	0.1396	0.0739	0.0397	0.0215

Table 3 illustrates the effect of the Hartmann number M – which represents the ratio of the magnetic Lorentz force to the viscous force, equation (18) – on the velocity profile for $M = 0, 2, 4$ and 6. For all $Y > 0$, the velocity decreases monotonically as M increases. With $M = 0$ (no applied magnetic field), the velocity at $Y = 1$ is 0.5189; increasing the magnetic field to $M = 6$ reduces this value to 0.2776, a reduction of about 46%. This trend is consistent with the physical role of the term $-A_3MU$ in equation (15): the Lorentz force generated by the interaction of the induced electric current with the applied magnetic field acts as a resistive body force opposing the motion of the conducting nanofluid, thereby retarding the flow and reducing the momentum boundary-layer thickness. This magnetic damping

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Vol. 01, Issue 2, June 2026, pp : 219-231

effect is the basis of several flow-control applications, including electromagnetic braking and the suppression of buoyancy-driven instabilities in liquid-metal and nanofluid systems.

Table 4. Effect of the nanoparticle volume fraction φ on the temperature profile θ(Y) at τ = 1, α = 0.7, M = 2, Pr = 6.2 (Cu–water)

Y	φ = 0.00	φ = 0.02	φ = 0.04	φ = 0.06
0.0	1.0000	1.0000	1.0000	1.0000
0.5	0.5934	0.6127	0.6319	0.6502
1.0	0.3520	0.3754	0.3990	0.4228
1.5	0.2086	0.2300	0.2522	0.2748
2.0	0.1237	0.1409	0.1593	0.1786
2.5	0.0733	0.0863	0.1006	0.1162
3.0	0.0435	0.0529	0.0636	0.0756

Table 4 shows the temperature profile $\theta(Y)$ at $\tau = 1$ for four values of the nanoparticle volume fraction, $\varphi = 0.00$ (pure water), 0.02, 0.04 and 0.06. At every value of $Y > 0$, the temperature increases as φ increases; for example, at $Y = 1$, θ increases from 0.3520 for pure water to 0.4228 for $\varphi = 0.06$, an increase of about 20%. This increase reflects the enhanced effective thermal diffusivity α_{nf} of the nanofluid: although the Maxwell model (equation (7)) shows that k_{nf} increases with φ, the effective heat capacitance $(\rho c_p)_{nf}$ decreases with φ because the heat capacity of the Cu nanoparticles is much lower than that of water (Table 1). The net effect is an increase in $\alpha_{nf} = k_{nf}/(\rho c_p)_{nf}$ with φ, which accelerates the diffusion of heat away from the heated plate and produces a thicker thermal boundary layer – that is, higher temperatures at a given Y and τ – for nanofluids with higher nanoparticle loading.

Table 5. Skin friction coefficient (Cf) and local Nusselt number (Nu) at τ = 1 for various α, M and φ (Cu–water)

α	M	φ	Cf	Nu
0.3	2	0.04	1.842	0.624
0.5	2	0.04	1.756	0.658
0.7	2	0.04	1.652	0.701
1.0	2	0.04	1.498	0.756

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Vol. 01, Issue 2, June 2026, pp : 219-231

0.7	0	0.04	1.214	0.701
0.7	4	0.04	1.978	0.701
0.7	2	0.00	1.598	0.612
0.7	2	0.02	1.624	0.658
0.7	2	0.06	1.683	0.745

Table 5 reports the skin friction coefficient *Cf* and the local Nusselt number *Nu*, defined by equation (19), for a range of values of α, *M* and φ, with the magnitudes of *Cf* reported (the velocity gradient at the wall is negative). Three trends are evident. First, *Cf* decreases as α increases from 0.3 to 1.0, consistent with the thickening of the momentum boundary layer with increasing α noted in Table 2 – a thicker boundary layer corresponds to a smaller velocity gradient at the wall – while *Nu* increases with α, indicating that the rate of heat transfer at the plate is also enhanced as the memory effect weakens. Second, *Cf* increases markedly with the Hartmann number *M*, from 1.214 at $M = 0$ to 1.978 at $M = 4$, because the Lorentz force confines the momentum boundary layer closer to the plate and steepens the velocity gradient at the wall, even though it reduces the velocity magnitude away from the wall (Table 3); *Nu* is essentially unaffected by *M* at this Prandtl number, since the magnetic field appears only in the momentum equation (15) and not directly in the energy equation (16). Third, both *Cf* and *Nu* increase with the nanoparticle volume fraction φ: the increase in *Cf* with φ is attributable to the increase in the effective viscosity μ_{nf} predicted by the Brinkman model (equation (5)), while the increase in *Nu* with φ reflects the enhanced effective thermal conductivity k_{nf} of the nanofluid, which is the principal motivation for the use of nanofluids in heat-transfer applications.

4.1 Comparison of Cu–Water and Al2O3–Water Nanofluids

Table 6. Comparison of skin friction coefficient (Cf) and local Nusselt number (Nu) for Cu–water and Al2O3–water at τ = 1, α = 0.7, M = 2

φ	Cf (Cu–water)	Nu (Cu–water)	Cf (Al2O3–water)	Nu (Al2O3–water)
0.00	1.598	0.612	1.598	0.612
0.02	1.624	0.658	1.611	0.641
0.04	1.652	0.701	1.626	0.668
0.06	1.683	0.745	1.642	0.694

Table 6 compares the skin friction coefficient and Nusselt number for Cu–water and Al_2O_3–water nanofluids over the same range of nanoparticle volume fraction, with all other parameters held fixed ($\alpha = 0.7$, $M = 2$, $Gr = 5$, $\tau = 1$). At $\varphi = 0$, the two nanofluids reduce to the same base fluid (water) and therefore give identical values of *Cf* and *Nu*. As φ increases, both *Cf* and *Nu* increase for both nanofluids, but the increase is consistently larger for Cu–water than for

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Al_2O_3–water; at φ = 0.06, Nu = 0.745 for Cu–water compared with Nu = 0.694 for Al_2O_3–water, an enhancement of about 7.3%. This difference is a direct consequence of the much higher thermal conductivity of Cu relative to Al_2O_3

(Table 1): for the same nanoparticle loading, the Maxwell model (equation (7)) predicts a larger increase in k_{nf} for Cu–water, which translates into a higher Nusselt number. The corresponding increase in Cf for Cu–water is associated with the slightly higher effective electrical conductivity and density of Cu, which marginally increase the magnetic retardation and the inertial contribution to the wall shear stress. These results indicate that, for applications where enhanced heat transfer is the primary requirement, Cu–water nanofluid offers a modest advantage over Al_2O_3–water at the same nanoparticle volume fraction, at the cost of a marginally higher wall shear stress.

5. Conclusion

A numerical investigation of unsteady magnetohydrodynamic free convective flow of Cu–water and Al_2O_3–water nanofluids past an impulsively started vertical plate has been carried out within the framework of fractional calculus, with the classical time derivatives in the momentum and energy equations replaced by the Caputo fractional derivative of order α. The resulting fractional partial differential equations were solved using a finite difference scheme combining the L1 approximation of the Caputo derivative with an implicit Crank–Nicolson spatial discretisation, and the scheme was validated against the classical integer-order solution. The principal conclusions drawn from the parametric study are summarised below.

1. The fractional order parameter α introduces a tunable memory effect: decreasing α from 1.0 toward 0.3 slows the development of both the velocity and temperature boundary layers, reducing U and (by implication) θ at a fixed dimensionless time τ, increasing the skin friction coefficient and reducing the Nusselt number.

2. Increasing the Hartmann number M substantially reduces the velocity field through the retarding Lorentz force, with the velocity at Y = 1 falling by about 46% as M increases from 0 to 6, while simultaneously increasing the skin friction coefficient because the momentum boundary layer is confined closer to the plate; the Nusselt number is essentially unaffected by M at the Prandtl number considered.

3. Increasing the nanoparticle volume fraction φ increases both the temperature field and the local Nusselt number, reflecting the enhanced effective thermal diffusivity and thermal conductivity of the nanofluid; it also produces a modest increase in the skin friction coefficient due to the increase in effective viscosity predicted by the Brinkman model.

4. For the same nanoparticle volume fraction, Cu–water nanofluid yields a higher Nusselt number than Al_2O_3–water (by approximately 7% at φ = 0.06), reflecting the substantially higher thermal conductivity of Cu, at the cost of a marginally higher skin friction coefficient.

5. The L1–Crank–Nicolson finite difference scheme developed in this work is stable, accurate to within 0.5% of the classical solution in the integer-order limit, and computationally efficient, making it suitable for the analysis of a wide range of fractional-order MHD nanofluid problems with general boundary and initial conditions.

Overall, the results demonstrate that the fractional order parameter provides an additional, physically meaningful degree of freedom for modelling nanofluids and other complex fluids whose momentum and thermal response exhibit memory-dependent (non-Markovian) behaviour, complementing the well-established effects of the magnetic field and nanoparticle loading. The present single-phase, one-dimensional model could be extended in future work to two- and

three-dimensional geometries, to two-phase (Buongiorno) nanofluid models that account for thermophoresis and Brownian diffusion of the nanoparticles, to variable-order fractional derivatives, and to other fractional operators such as the Caputo–Fabrizio and Atangana–Baleanu derivatives, in order to further generalise the memory-dependent description of MHD nanofluid transport.

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